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Compact spaces, elementary submodels, and the countable chain condition, II

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Abstract

Given a space $\langle X, \mathcal{T}\rangle$ in an elementary submodel of $H(\theta)$, define X_M to be $X \cap M$ with the topology generated by $\{U \cap M\colon U \in \mathcal{T} \cap M\}$. It is established that if X_M is compact and satisfies the countable chain condition, while X is not scattered and has cardinality less than the first inaccessible cardinal, then $X = X_M$. If the character of X_M is a member of M, then "inaccessible" may be replaced by "1-extendible".

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This paper continues the line of research of [4,6,9–12], in which the question of which topological spaces are determined by their compact reflections in elementary submodels is investigated.

Given a space $\langle X, \mathcal{T}\rangle$ in an elementary submodel of $H(\theta)$, we define X_M to be $X \cap M$ with the topology generated by $\{U \cap M\colon U \in \mathcal{T} \cap M\}$. See [5] for the basic properties of such X_M's. Since $H(\theta)$ is supposed to be a stand-in for the universe V (see e.g. [7, Chapter 24]), assume θ is regular and "sufficiently large". As a concrete manifestation of largeness, we will consider θ to have cardinality greater than all finite iterations of the power-set function starting with X.

If X_M is compact T_2 (in fact, we shall assume all spaces are T_2), this constrains X to the point that simple additional topological hypotheses on X_M ensure that $X_M = X$ [6]. When powers of the two-point discrete space D are considered, the situation is more complicated: roughly, for κ below very large cardinals, X_M homeomorphic to D^κ implies $X_M = X$, but this is not the case above such large cardinals [6,9,11]. This was generalized to continuous images of powers of D (dyadic compacta) in [12] and to compact spaces with regular open algebras isomorphic to those of dyadic compacta in [4]. In [6], the question of whether a generalization to compact spaces satisfying the countable chain condition was consistent was raised. In [4] it was proved, assuming the Singular Cardinals Hypothesis and the negation of a weak version of Chang's Conjecture, that if X was not scattered, X_M was compact and satisfied the

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countable chain condition, and the character of X_M was in M, then $X_M = X$. This result was somewhat unsatisfying in that it was not clear whether the Singular Cardinals Hypothesis was needed, or whether this result really needed more "anti-Chang" power than the D^κ results. In this paper we remedy both defects:

Theorem 1. *If X is not scattered and has cardinality less than the first inaccessible, while X_M is compact and satisfies the countable chain condition, then $X = X_M$. If the character of X_M is in M, "inaccessible" can be replaced by "1-extendible".*

Rather weak set-theoretic assumptions (e.g. the non-existence of $0^\#$) imply there are no 1-extendible cardinals. We refer the reader to [9] or [8] for the definition of 1-extendibility, since we will not use it. Suffice it to say that a 1-extendible cardinal λ is inaccessible and in fact is the λth measurable cardinal.

The proof will use ingredients similar to those in [4], but packaged somewhat differently. The following definition is convenient.

Definition. [9] A compact space X is *squashable* if there is an M containing X such that X_M is compact and not equal to X.

Although X_M can be compact without equalling X—see [6] for several examples—its compactness does entail the compactness of X:

Lemma 2. [3] *If X_M is compact, so is X, and there is a perfect map from X onto X_M.*

Kunen [9] improved previous work [6,11] to get the following result:

Lemma 3. [9] *If κ is less than or equal to the first* 1*-extendible cardinal, then D^κ is not squashable.*

A key concept in Kunen's work is the following:

Definition. A λ-*Čech–Pospíšil tree* in a space X is a tree $\mathcal{K} = \{K_s\colon s \in {}^{\leqslant\lambda}2\}$ satisfying:

(1) Each K_s is non-empty and closed in X.
(2) $s \subseteq t$ implies $K_s \supseteq K_t$.
(3) $K_{s^\frown 0} \cap K_{s^\frown 1} = \emptyset$.
(4) If the length of $s = \gamma$, a limit ordinal, then $K_s = \bigcap_{\alpha<\gamma} K_{s|\alpha}$.

Definitions of the standard cardinal invariants we shall be using appear in [1,2]. In addition, let us make the following definitions:

Definition. $\mathrm{sc}(X)$, the *scattering number* of X, is the least infinite cardinal κ such that for each closed subset F of X, there is an $x \in F$ with $\chi(x, F) < \kappa$. $m(X)$, the *mapping number* of X, is the least infinite cardinal κ such that for each closed subset F of X, there is an $x \in F$ with $\pi\chi(x, F) < \kappa$.

The reason for the name of $m(X)$ will be evident from Lemma 6 below. Čech and Pospíšil (see e.g. [2, 3.16]) proved:

Proposition 4. *If X is compact and $\chi(p, X) \geqslant \lambda$ for all $p \in X$, then there is a λ-Čech–Pospíšil tree in X and hence $|X| \geqslant 2^\lambda$.*

Kunen translated this into the language of submodels; a reformulation of his work yields:

Lemma 5. [4] *If X_M is compact, $\lambda < \mathrm{sc}(X)$ and $\lambda + 1 \subseteq M$, then $2^\lambda \subseteq M$.*

Here is a different array of closed sets, and an important result of Šapirovskiĭ (see e.g. [2, 3.18]):

Definition. [2] A κ*-dyadic system* in a space X is a family $\{\langle F^0_\alpha, F^1_\alpha\rangle\colon \alpha < \kappa\}$ of pairs of non-empty closed subsets of X such that:

(a) $F^0_\alpha \cap F^1_\alpha = 0$ for all $\alpha < \kappa$,
(b) $F_\varepsilon = \bigcap\{F^{\varepsilon(\alpha)}_\alpha\colon \alpha \in \operatorname{dom}\varepsilon\}$, for each finite partial function ε from κ to 2.

Lemma 6. *The following conditions are equivalent for compact T_2 spaces*:

(i) *X can be mapped continuously onto I^κ,*
(ii) *there is a closed $F \subseteq X$ which can be mapped onto D^κ,*
(iii) *there is a closed $F \subseteq X$ with $\pi\chi(x, F) \geqslant \kappa$ for each $x \in F$,*
(iv) *there is a κ-dyadic system in X.*

Using Lemma 6, we can vary Lemma 5 to obtain:

Lemma 7. [6, essentially] *If X_M is compact, $\lambda \in M$, $\lambda < m(X)$, and D^λ is not squashable, then $2^\lambda \subseteq M$.*

We give the short, instructive proof. Since $\lambda < m(X)$, take $F \in M$, a closed subset of X, and $f \in M$ mapping F onto D^λ. Then f restricted to $F \cap M$ maps F_M onto $(D^\lambda)_M$, so $(D^\lambda)_M$ is compact. But then $(D^\lambda)_M = D^\lambda$ and so $2^\lambda \subseteq M$.

It is sometimes useful to draw conclusions about the existence of nice dense sets from information about $\mathrm{sc}(X)$ or $m(X)$. Recall $D \subseteq X$ is G_δ*-dense* if D meets every non-empty G_δ subset of X. The following result comes from [4,12], but is likely folklore.

Lemma 8. *Suppose X is compact.*

(a) *If $\mathrm{sc}(X) = \kappa$ is uncountable, then $\{x\colon \chi(x, X) < \kappa\}$ is G_δ-dense in X. If $\mathrm{sc}(X) = \aleph_0$, then $\{x\colon \chi(x, X) \leqslant \aleph_0\}$ is G_δ-dense in X.*
(b) *If $m(X) = \kappa$ is uncountable, then $\{x\colon \pi\chi(x, X) < \kappa\}$ is G_δ-dense in X. If $m(X) = \aleph_0$, then $\{x\colon \chi(x, X) \leqslant \aleph_0\}$ is G_δ-dense in X.*

Proof. In a compact space, every non-empty G_δ set G includes a non-empty compact G_δ, say K_G. Take $x \in K_G$ with $\chi(x, K_G)$ (respectively, $\pi\chi(x, K_G) < \kappa$. Then $\chi(x, X) \leqslant \chi(x, K_G) \cdot \chi(K_G, X)$. Similarly, $\pi\chi(x, X) \leqslant \pi\chi(x, K_G) \cdot \chi(K_G, X)$. Since $\chi(K_G, X) \leqslant \aleph_0$, an easy calculation gives the claimed results. □

It is useful to calculate how $\mathrm{sc}(X)$ and $m(X)$ relate to some of the other cardinal invariants of X:

Lemma 9.

(a) $m(X) \leqslant \mathrm{sc}(X)$;
(b) $\mathrm{sc}(X) \leqslant w(X)^+$;
(c) *For X compact,* $\mathrm{sc}(X) \leqslant |X|^+$;
(d) *For X regular,* $w(X) \leqslant \pi\chi(X)^{c(X)}$;
(e) *For X regular,* $w(X) \leqslant m(X)^{c(X)}$.

Proof. The first is clear from the definitions. The second is obvious, since if $w(X) = \lambda$, each closed set has a point of character less than λ^+. The third is because $w(X) \leqslant |X|$ for compact spaces. The fourth is a well-known result of Šapirovskiĭ—see e.g. [1]. For the fifth, by Lemma 8, X has a dense set D of points, each of π-character less than $m(X)$. Then $\pi w(D) = \pi w(X)$, $c(D) = c(X)$, and by (d), $\pi w(D) \leqslant \pi\chi(D)^{c(D)}$. For $x \in D$, $\pi\chi(x, D) = \pi\chi(x, X)$, so $\pi\chi(D) \leqslant m(X)$. Then $\pi w(D) \leqslant m(X)^{c(D)}$, so $\pi w(X) \leqslant m(X)^{c(X)}$, and therefore $w(X) \leqslant m(X)^{c(X)}$. □

We now have most of the ingredients needed to prove Theorem 1. In order to prove $X = X_M$, we will quote a result from [6]:

Lemma 10. *Let $X \in M$ be compact. If $\chi(X) \subseteq M$, then $X = X_M$.*

Proof of Theorem 1. There are several cases to consider, depending on what kind of a cardinal $m(X) = \kappa$ is. Note that if $|X| <$ the first 1-extendible, so is $m(X)$.

Case 1. $\kappa \leqslant 2^{\aleph_0}$.

This will be excluded by Lemma 3 and the following result. Recall a space is *scattered* if each subspace has a point isolated in it.

Lemma 11. [2, 3.17] *If X is compact and is not scattered, then $m(X) > \aleph_0$.*

A useful corollary of this is:

Lemma 12. *If X_M is compact and satisfies the countable chain condition, and X is not scattered, then X satisfies the countable chain condition.*

Proof. As observed in [6], it suffices to prove that $\omega_1 \subseteq M$ which follows from Lemmas 2, 3, 11 and 7. □

Now if $\kappa \leqslant 2^{\aleph_0}$, $\chi(X) \leqslant w(X) \leqslant \kappa^{\aleph_0} \leqslant 2^{\aleph_0} \subseteq M$ (by the lemmas in the previous line), so Case 1 is established.

Case 2. $\kappa > 2^{\aleph_0}$, and there is a $\lambda < \kappa$ such that $2^{\lambda} \geqslant \kappa$.

Since X and hence $\kappa \in M$, there is such a $\lambda \in M$. Then by Lemmas 3 and 7, $2^{\lambda} \subseteq M$. $\kappa^{\aleph_0} \leqslant 2^{\lambda}$, so $w(X) \subseteq M$ by Lemma 9 and $X = X_M$ by Lemma 10.

Case 3. κ is a strong limit, $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$.

As before, $w(X) \leqslant \kappa^{\aleph_0} = \kappa$. Let $\lambda = \mathrm{cf}(\kappa)$. $\lambda \in M$ and we may take $\{\lambda_\alpha\}_{\alpha<\lambda}$ in M converging to κ. Since $\lambda < \kappa$, by Lemma 7, 2^{λ} and hence $\lambda \subseteq M$. But then each $\lambda_\alpha \in M$, so $2^{\lambda_\alpha} \subseteq M$, since $2^{\lambda} \subseteq M$. But then $2^{<\kappa}$ and hence $\kappa \subseteq M$ and we are done.

Case 4. $\mathrm{cf}(\kappa) = \omega$, κ a strong limit.

This is the delicate case, and is the only one in which we need to consider character as well as π-character. By the same proof as for Case 3, note we can conclude that $\kappa \subseteq M$. If $\mathrm{sc}(X) > \kappa$, then by Lemma 5, $2^{\kappa} \subseteq M$. Now by Lemma 9, $w(X) \leqslant \kappa^{\aleph_0} \leqslant 2^{\kappa}$, so we are done by Lemma 10. If $\mathrm{sc}(X) = \kappa$, we proceed as in [4], letting $\{\kappa_n\}_{n<\omega}$ be an increasing sequence in M of uncountable cardinals in M converging to κ. Let $E_n = \{x \in X\colon \chi(x, X) < \kappa_n\}$. Let $F_n = \overline{E}_n$. Claim $X = \bigcup_{n<\omega} F_n$. For if not, $X - \bigcup_{n<\omega} F_n = \bigcap_{n<\omega}(X - F_n) \neq 0$ and would have to intersect $\bigcup_{n<\omega} E_n$ by Lemma 8. It thus will suffice to prove that $(F_n)_M = F_n$, for every n.

The reason we need to use character rather than π-character here is the following observation from [2, p. 21] used in [4]:

Lemma 13. $|\{p \in Y\colon \chi(p, Y) < \lambda|\} \leqslant 2^{c(Y)\cdot\lambda}$.

Thus $|E_n| \leqslant 2^{\aleph_0\cdot\kappa_n} = 2^{\kappa_n}$. Since $w(Z) \leqslant 2^{d(Z)}$ for regular Z, $w(F_n) \leqslant 2^{2^{\kappa_n}} < \kappa$. Since $\kappa \subseteq M$, each $(F_n)_M = F_n$, and hence $X_M = X$.

We can prove the second part of Theorem 1 by closer analysis. Observe that the proof of Case 3 would work for inaccessible $\kappa = m(X)$ if we knew that $|M \cap \kappa| = \kappa$, for then $2^{<\kappa} = \Sigma\{2^{\mu}\colon\ \mu < \kappa,\ \mu \in M\}$. If $\kappa = m(X)$ is inaccessible, then $\kappa^{\aleph_0} = \kappa$. Let $\lambda = \chi(X_M)$. If $\lambda < \kappa$, then as usual, we conclude that $2^\lambda \subseteq M$. In this case $X = X_M$ by the following result from [12], generalizing [6]:

Lemma 14. *If X_M is compact, $\chi(X_M) \leqslant \mu$ and $2^\mu \subseteq M$, then $X = X_M$.*

Suppose then that $\kappa \leqslant \lambda$. Then applying Lemma 9, we have:

$$\kappa \leqslant \lambda \leqslant \left|M \cap w(X)\right| \leqslant \left|M \cap \kappa^{\aleph_0}\right| = |M \cap \kappa|.$$

Hence $|M \cap \kappa| = \kappa$ and we are done. □

Remarks. Theorem 1 is a considerable improvement over [4], where the Singular Cardinals Hypothesis and the negation of a form of Chang's Conjecture were needed to draw essentially the same conclusion. Both that paper's results and ours use the annoying hypothesis that $\chi(X_M) \in M$. "Character" is not special; e.g. $d(X_M)$ or $w(X_M)$ would also work. In fact, such hypotheses cannot be completely dropped. It is well known that a measurable cardinal—and, in particular a 1-extendible cardinal—has a weakly compact cardinal below it. Kunen [9] showed:

Lemma 15. *Let X be compact and assume $\kappa = |X|$ is weakly compact. Then X is squashable.*

Now notice that if X satisfies the countable chain condition, so does X_M, since if X_M admits an uncountable disjoint collection of open sets, it admits one of basic open sets. But if $\{U_\alpha \cap M\colon \alpha < w_1\}$ are disjoint, where $U_\alpha \in M$ is open, by elementarity $U_\alpha \cap U_\beta = 0$, for $\alpha \neq \beta$. Thus, all we need to do to construct the following example is to find a compact X of size κ satisfying the countable chain condition.

Example. A squashable countable chain condition space of size less than the first 1-extendible.

This is easy to do: just let $X = \prod_{\lambda<\kappa} D^\lambda$, where κ is weakly compact and less than the first 1-extendible. X is compact, and by inaccessibility, $|X| = \kappa$. X is a product of products of separable spaces, and so satisfies the countable chain condition.

The bound given by the 1-extendible cardinal in Theorem 1 can actually be slightly increased—see [9]. The requirement in Theorem 1 that X not be scattered cannot be removed: in [6] it is proved that every uncountable compact scattered space is squashable. We could however replace it by requiring that X_M not be scattered. To see this, apply Lemmas 2 and 11 to conclude that this would also imply $m(X) > \aleph_0$.

There are a number of ways of packaging the content of the proof of Theorem 1. Here is another one:

Corollary 16. *Suppose there is a non-scattered compact space X which can be squashed to a compact X_M satisfying the countable chain condition and having $\chi(X_M) \in M$. Then for some $\lambda < m(X)$, D^λ is squashable.*

As in [4], we can partially translate our results into the language of Boolean algebras. For example, we have:

Theorem 17. *Suppose $B \in M$ is a countable chain condition, atomless Boolean algebra such that (Stone space of $B)_M$ is compact, and $|B| <$ the first inaccessible. Then $B \subseteq M$.*

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